

Vector Meson Dominance as a first step in a systematic approximation: the pion vector form factor

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Abstract

Padé Approximants can be used to go beyond Vector Meson Dominance in a systematic approximation. We illustrate this fact with the case of the pion vector form factor and extract values for the first two coefficients of its Taylor expansion. Padé Approximants are shown to be a useful and simple tool for incorporating high-energy information, allowing an improved determination of these Taylor coefficients.

1 Introduction

It has been known for a long time that the pion vector form factor (VFF) in the space-like region is very well described by a monopole ansatz of the type given by Vector Meson Dominance (VMD) in terms of the rho meson. However, it has remained unclear whether there is a good reason for this from QCD or it is just a mere coincidence and, consequently, it is not known how to go about improving on this ansatz.

To begin our discussion, let us define the form factor, $F(Q^2)$, by the matrix element

$$\langle \pi^+(p') | \frac{2}{3} \bar{u} \gamma^\mu u - \frac{1}{3} \bar{d} \gamma^\mu d - \frac{1}{3} \bar{s} \gamma^\mu s | \pi^+(p) \rangle = (p + p')^\mu F(Q^2), \quad (1)$$

where $Q^2 = -(p' - p)^2$, such that $Q^2 > 0$ corresponds to space-like data. Since the spectral function for the corresponding dispersive integral for $F(Q^2)$ starts at twice the pion mass, the form factor can be approximated by a Taylor expansion in powers of the momentum for $|Q^2| < (2m_\pi)^2$. At low momentum, Chiral Perturbation Theory is the best tool for organizing the pion interaction in a systematic expansion in powers of momenta and quark masses [1, 2, 3]. With every order in the expansion, there comes a new set of coupling constants, the so-called low-energy constants (LECs), which encode all the QCD physics from higher energies. This means, in particular, that the coefficients in the Taylor expansion can be expressed in terms of these LECs and powers of the quark masses. Consequently, by learning about the low-energy expansion, one may indirectly extract important information about QCD.

In principle, the coefficients in the Taylor expansion may be obtained by means of a polynomial fit to the experimental data in the space-like region ¹ below $Q^2 = 4m_\pi^2$. However, such a polynomial fit implies a tradeoff. Although, in order to decrease the (systematic) error of the truncated Taylor expansion, it is clearly better to go to a low-momentum region, this also downsizes the set of data points included in the fit which, in turn, increases the (statistical) error. In order to achieve a smaller statistical error one would have to include experimental data from higher energies, i.e. from $Q^2 > 4m_\pi^2$. Since this is not possible in a polynomial fit, the use of alternative mathematical descriptions may be a better strategy.

One such description, which includes time-like data as well, is based on the use of the Roy equations and Omnés dispersion relations. This is the avenue followed by [4, 5], which has already produced interesting results on the scalar channel [6], and which can also be applied to the vector channel. Other procedures have relied on conformal transformations for the joint analysis of both time-like and space-like data [7], or subtracted Omnés relations [8, 9]. Further analyses may be found in Ref. [10].

On the other hand, as already mentioned above, one may also consider an ansatz of the type

$$F(Q^2)_{\text{VMD}} = \left(1 + \frac{Q^2}{M_{\text{VMD}}^2} \right)^{-1}. \quad (2)$$

¹Time-like data is provided by $\pi\pi$ production experiments and, consequently, they necessarily correspond to values of the momentum above the $\pi\pi$ cut, i.e. $|Q^2| > 4m_\pi^2$ with $Q^2 < 0$.

Even though the simplicity of the form of Eq. (2) is quite astonishing, it reproduces the space-like data rather well, even for a range of momentum of the order of a few GeV, i.e. $Q^2 \gg 4m_\pi^2$. If this fact is not merely a fluke, it could certainly be interesting to consider the form (2) as the first step in a systematic approximation, which would then allow improvement on this VMD ansatz.

In this article, we would like to point out that the previous VMD ansatz for the form factor (2) can be viewed as the first element in a sequence of Padé Approximants (PAs) which can be constructed in a systematic way. By considering higher-order terms in the sequence, one may be able to describe the space-like data with an increasing level of accuracy ². Of course, whether this is actually the case and the sequence is a convergent one in the strict mathematical sense or, on the contrary, the sequence eventually diverges, remains to be seen. But the important difference with respect to the traditional VMD approach is that, as a Padé sequence, the approximation is well-defined and can be systematically improved upon.

Although polynomial fitting is more common, in general, rational approximants (i.e. ratios of two polynomials) are able to approximate the original function in a much broader range in momentum than a polynomial [11]. This will be the great advantage of the Padés compared to other methods: they allow the inclusion of low and high energy information in a rather simple way which, furthermore, can in principle be systematically improved upon. In certain cases, like when the form factor obeys a dispersion relation given in terms of a positive definite spectral function (i.e. becomes a Stieltjes function), it is known that the Padé sequence is convergent everywhere on the complex plane, except on the physical cut [12]. Another case of particular interest is in the limit of an infinite number of colors in which the form factor becomes a meromorphic function. In this case there is also a theorem which guarantees convergence of the Padé sequence everywhere in a compact region of the complex plane, except perhaps at a finite number of points (which include the poles in the spectrum contained in that region) [13]. In the real world, in which a general form factor has a complicated analytic structure with a cut, and whose spectral function is not positive definite, we do not know of any mathematical result assuring the convergence of a Padé sequence [14]. One just has to try the approximation on the data to learn what happens.

In this work we have found that, to the precision allowed by the experimental data, there are sequences of PAs which improve on the lowest order VMD result in a rather systematic way. This has allowed us to extract the values of the lowest-order coefficients of the low-energy expansion.

We would like to emphasize that, strictly speaking, the Padé Approximants to a given function are ratios of two polynomials $P_N(z)$ and $Q_M(z)$ (with degree N and M , respectively), constructed such that the Taylor expansion around the origin exactly coincides with that of $f(z)$ up to the highest possible order, i.e. $f(z) - R_M^N(z) = \mathcal{O}(z^{M+N+1})$. However, in our case the Taylor coefficients are not known. They are, in fact, the information we are seeking for. Our strategy will consist in determining these coefficients by a least-squares fit of a Padé Approximant to the vector form factor data

²Obviously, unlike the space-like data, one should not expect to reproduce the time-like data since a Padé Approximant contains only isolated poles and cannot reproduce a time-like cut.

in the space-like region.

There are several types of PAs that may be considered. In order to achieve a fast numerical convergence, the choice of which one to use is largely determined by the analytic properties of the function to be approximated. In this regard, a glance at the time-like data of the pion form factor makes it obvious that the form factor is clearly dominated by the rho meson contribution. The effect of higher resonance states, although present, is much more suppressed. In these circumstances the natural choice is a P_1^L Padé sequence [11], i.e. the ratio of a polynomial of degree L over a polynomial of degree one ³. Notice that, from this perspective, the VMD ansatz in (2) is nothing but the P_1^0 Padé Approximant.

However, to test the aforementioned single-pole dominance, one should check the degree to which the contribution from resonances other than the rho may be neglected. Consequently, we have also considered the sequence P_2^L , and the results confirm those found with the PAs P_1^L . Furthermore, for completeness, we have also considered the so-called Padé-Type approximants (PTs) [15, 16]. These are rational approximants whose poles are predetermined at some fixed values, which we take to be the physical masses since they are known. Notice that this is different from the case of the ordinary PAs, whose poles are left free and determined by the fit. Finally, we have also considered an intermediate case, the so-called Partial-Padé approximants (PPs) [15], in which some of the poles are predetermined (again at the physical masses) and some are left free. We have fitted all these versions of rational approximants to all the available pion VFF space-like data [17]-[22]. The result of the fit is rather independent of the kind of rational approximant sequence used and all the results show consistency among themselves.

The structure of this letter is the following. In section 2 we begin by testing the efficiency of the P_1^L Padés with the help of a model. In section 3 we apply this very same method to the experimental VFF. Firstly, in sec. 3.1, we use the Padé Approximants P_1^L ; then, in Sec. 3.2, this result is cross-checked with a P_2^L PA. Finally, in sec. 3.3, we study the Padé-Type and Partial-Padé approximants. The outcome of these analyses is combined in section 4 and some conclusions are extracted.

2 A warm-up model

In order to illustrate the usefulness of the PAs as fitting functions in the way we propose here, we will first use a phenomenological model as a theoretical laboratory to check our method. Furthermore, the model will also give us an idea about the size of possible systematic uncertainties.

We will consider a VFF phase-shift with the right threshold behavior and with approximately the physical values of the rho mass and width. The form-factor is

³Conventionally, without loss of generality, the polynomial in the denominator is normalized to unity at the origin.

recovered through a once-subtracted Omnés relation,

$$F(Q^2) = \exp \left\{ -\frac{Q^2}{\pi} \int_{4\hat{m}_\pi^2}^{\infty} dt \frac{\delta(t)}{t(t+Q^2)} \right\} , \quad (3)$$

where $\delta(t)$, which plays the role of the vector form factor phase-shift [8, 9, 23], is given by

$$\delta(t) = \tan^{-1} \left[\frac{\hat{M}_\rho \hat{\Gamma}_\rho(t)}{\hat{M}_\rho^2 - t} \right] , \quad (4)$$

with the t -dependent width given by

$$\hat{\Gamma}_\rho(t) = \Gamma_0 \left(\frac{t}{\hat{M}_\rho^2} \right) \frac{\sigma^3(t)}{\sigma^3(\hat{M}_\rho^2)} \theta(t - 4\hat{m}_\pi^2) , \quad (5)$$

and $\sigma(t) = \sqrt{1 - 4\hat{m}_\pi^2/t}$. The input parameters are chosen to be close to their physical values:

$$\Gamma_0 = 0.15 \text{ GeV} , \quad \hat{M}_\rho^2 = 0.6 \text{ GeV}^2 , \quad 4\hat{m}_\pi^2 = 0.1 \text{ GeV}^2 . \quad (6)$$

We emphasize that the model defined by the expressions (3-5) should be considered as quite realistic. In fact, it has been used in Ref. [8, 9, 23] for extracting the values for the physical mass and width of the rho meson through a direct fit to the (timelike) experimental data.

Expanding $F(Q^2)$ in Eq. (3) in powers of Q^2 we readily obtain

$$F(Q^2) = 1 - a_1 Q^2 + a_2 Q^4 - a_3 Q^6 + \dots , \quad (7)$$

with known values for the coefficients a_i . In what follows, we will use Eq. (7) as the definition of the coefficients a_i . To try to recreate the situation of the experimental data [17]-[22] with the model, we have generated fifty “data” points in the region $0.01 \leq Q^2 \leq 0.25$, thirty data points in the interval $0.25 \leq Q^2 \leq 3$, and seven points for $3 \leq Q^2 \leq 10$ (all these momenta in units of GeV^2). These points are taken with vanishing error bars since our purpose here is to estimate the systematic error derived purely from our approximate description of the form factor.

We have fitted a sequence of Padé Approximants $P_1^L(Q^2)$ to these data points and, upon expansion of the Padés around $Q^2 = 0$, we have used them to predict the values of the coefficients a_i . The comparison may be found in Table 1. The last PA we have fitted to these data is P_1^6 . Notice that the pole position of the Padés differs from the true mass of the model, given in Eq. (6).

A quick look at Table 1 shows that the sequence seems to converge to the exact result, although in a hierarchical way, i.e. much faster for a_1 than for a_2 , and this one much faster than a_3 , etc... The relative error achieved in determining the coefficients a_i by the last Padé, P_1^6 , is respectively 0.9%, 8% and 26% for a_1, a_2 and a_3 . Naively, one would expect these results to improve as the resonance width decreases since the P_1^L contains only a simple pole, and this is indeed what happens. Repeating this exercise with the model, but with a $\Gamma_0 = 0.015 \text{ GeV}$ (10 times smaller than the previous one),

	P_1^0	P_1^1	P_1^2	P_1^3	P_1^4	P_1^5	P_1^6	$F(Q^2)(\text{exact})$
$a_1(\text{GeV}^{-2})$	1.549	1.615	1.639	1.651	1.660	1.665	1.670	1.685
$a_2(\text{GeV}^{-4})$	2.399	2.679	2.809	2.892	2.967	3.020	3.074	3.331
$a_3(\text{GeV}^{-6})$	3.717	4.444	4.823	5.097	5.368	5.579	5.817	7.898
$s_p(\text{GeV}^2)$	0.646	0.603	0.582	0.567	0.552	0.540	0.526	0.6

Table 1: Results of the various fits to the form factor $F(Q^2)$ in the model, Eq. (3). The exact values for the coefficients a_i in Eq. (7) are given on the last column. The last row shows the predictions for the corresponding pole for each Padé (s_p), to be compared to the true mass $\hat{M}_\rho^2 = 0.6 \text{ GeV}^2$ in the model.

the relative error achieved by P_1^6 for the same coefficients as before is 0.12%, 1.1% and 4.7%. On the other hand, a model with Γ_0 five times bigger than the first one produces, respectively, differences of 2.1%, 14.4% and 37.8%.

As we have mentioned in the introduction, it is possible to build a variation of the PAs, the Padé-Type Approximants, where one fixes the pole in the denominator at the physical mass and only the numerator is fitted. We have also studied the convergence of this kind of rational approximant with the model. Thus, in this case, we have placed the P_1^L pole at $s_p = \hat{M}_\rho^2$ and found a similar pattern as in Table 1. For P_1^6 , the Padé-Type coefficient a_1 differs a 2.5% from its exact value, a_2 by 16% and a_3 by 40%.

Based on the previous results, we will take the values in Table 1 as a rough estimate of the systematic uncertainties when fitting to the experimental data in the following sections. Since, as we will see, the best fit to the experimental data comes from the Padé P_1^4 , we will take the error in Table 1 from this Padé as a reasonable estimate and add to the final error an extra systematic uncertainty of 1.5% and 10% for a_1 and a_2 (respectively).

3 The pion vector form factor

We will use all the available experimental data in the space-like region, which may be found in Refs. [17]-[22]. These data range in momentum from $Q^2 = 0.015$ up to 10 GeV^2 .

As discussed in the introduction, the prominent role of the rho meson contribution motivates that we start with the P_1^L Padé sequence.

3.1 Padé Approximants P_1^L

Without any loss of generality, a P_1^L Padé is given by

$$P_1^L(Q^2) = 1 + \sum_{k=0}^{L-1} a_k (-Q^2)^k + (-Q^2)^L \frac{a_L}{1 + \frac{a_{L+1}}{a_L} Q^2}, \quad (8)$$

where the vector current conservation condition $P_1^L(0) = 1$ has been imposed and the coefficients a_k are the low-energy coefficients of the corresponding Taylor expansion of

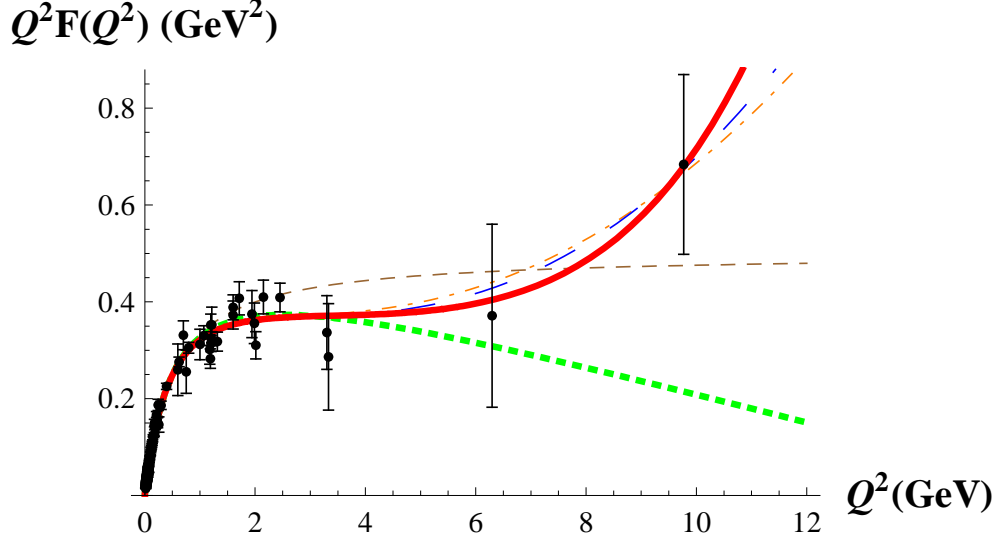


Figure 1: The sequence of P_1^L PAs is compared to the available space-like data [17]-[22]: P_1^0 (brown dashed), P_1^1 (green thick-dashed), P_1^2 (orange dot-dashed), P_1^3 (blue long-dashed), P_1^4 (red solid).

the VFF (compare with (7) for the case of the model in the previous section).

The fit of P_1^L to the space-like data points in Refs. [17]-[22] determines the coefficients a_k that best interpolate them. According to Ref. [24], the form factor is supposed to fall off as $1/Q^2$ (up to logarithms) at large values of Q^2 . This means that, for any value of L , one may expect to obtain a good fit only up to a finite value of Q^2 , but not for asymptotically large momentum. This is clearly seen in Fig. 1, where the Padé sequence P_1^L is compared to the data up to $L = 4$.

Fig. 2 shows the evolution of the fit results for the Taylor coefficients a_1 and a_2 for the P_1^L PA from $L = 0$ up to $L = 4$. As one can see, after a few Padés these coefficients become stable. Since the experimental data have non zero error it is only possible to fit a P_1^L PA up to a certain value for L . From this order on, the large error bars in the highest coefficient in the numerator polynomial make it compatible with zero and, therefore, it no longer makes sense to talk about a new element in the sequence. For the data in Refs. [17]-[22], this happened at $L = 4$ and this is why our plots stop at this value. Therefore, from the PA P_1^4 we obtain our best fit and, upon expansion around $Q^2 = 0$, this yields

$$a_1 = 1.92 \pm 0.03 \text{ GeV}^{-2}, \quad a_2 = 3.49 \pm 0.26 \text{ GeV}^{-4}; \quad (9)$$

with a $\chi^2/\text{dof} = 117/90$.

Eq. (8) shows that the pole of each P_1^L PA is determined by the ratio $s_p = a_L/a_{L+1}$. This ratio is shown in Fig. 3, together with a gray band whose width is given by $\pm M_\rho \Gamma_\rho$ for comparison. From this figure one can see that the position of the pole of the PA is close to the physical value M_ρ^2 [25], although it does not necessarily agree with it, as we already saw in the model of the previous section.

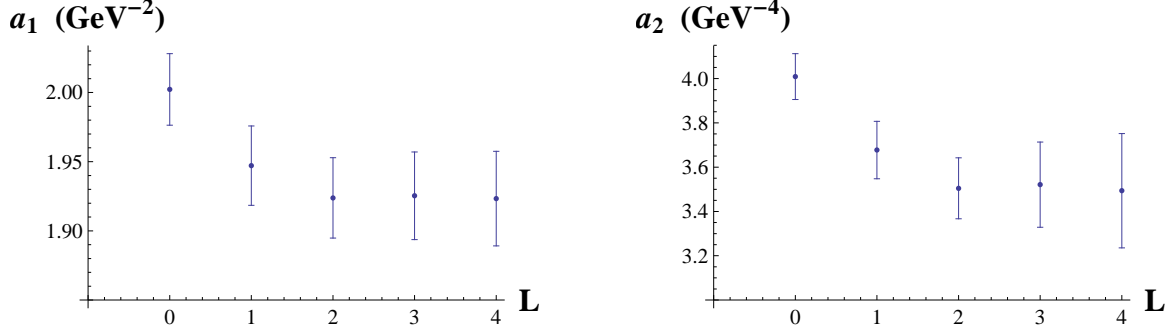


Figure 2: a_1 and a_2 Taylor coefficients for the P_1^L PA sequence.

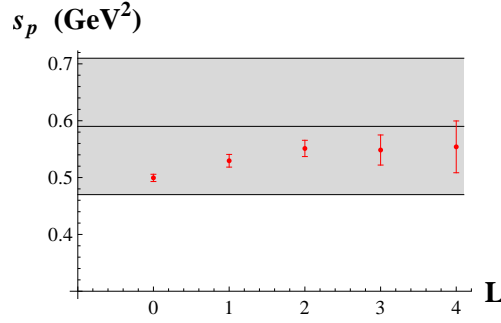


Figure 3: Position s_p of the pole for the different P_1^L . The range with the physical values $M_\rho^2 \pm M_\rho \Gamma_\rho$ is shown (gray band) for comparison.

3.2 Comment on P_2^L Padés

Although the time-like data of the pion form factor is clearly dominated by the $\rho(770)$ contribution, consideration of two-pole P_2^L PAs will give us a way to assess any possible systematic bias in our previous analysis, which was limited to only single-pole PAs.

We have found that the results of the fits of P_2^L PAs to the data tend to reproduce the VMD pattern found for the P_1^L PAs in the previous section. The P_2^L PAs place the first of the two poles around the rho mass, while the second wanders around the complex momentum plane together with a close-by zero in the numerator. This association of a pole and a close-by zero is what is called a “defect” in the mathematical literature[26]. A defect is only a local perturbation and, at any finite distance from it, its effect is essentially negligible. This has the net effect that the P_2^L Padé in the euclidean region looks just like a P_1^L approximant and, therefore, yields essentially the same results. For example, for the P_2^2 , one gets

$$a_1 = 1.924 \pm 0.029 \text{ GeV}^{-2}, \quad a_2 = 3.50 \pm 0.14 \text{ GeV}^{-4}, \quad (10)$$

with a $\chi^2/\text{dof} = 120/92$.

3.3 Padé Type and Partial Padé Approximants

Besides the ordinary Padé Approximants one may consider other kinds of rational approximants. These are the Padé Type and Partial Padé Approximants [13, 15, 16].

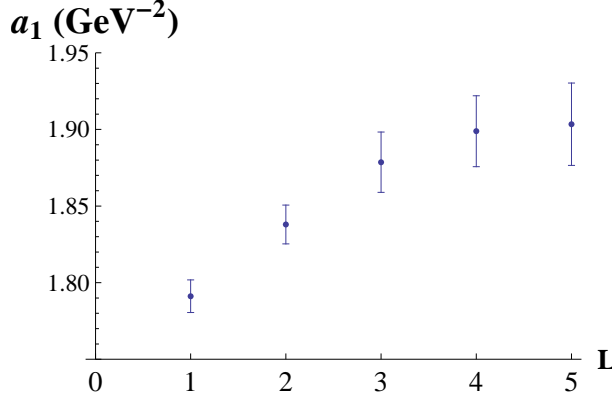


Figure 4: Low-energy coefficient a_1 from the T_1^L Padé-Type sequence

In the Padé Type Approximants (PTAs) the poles of the Padé are fixed to certain particular values, which in this context are naturally the physical masses. On the other hand, in the Partial Padé Approximants (PPAs) one has an intermediate situation between the PAs and the PTAs in which some poles are fixed while others are left as free parameters to fit.

Since the value of the physical rho mass is known ($M_\rho = 775.5$ MeV), it is natural to attempt a fit of PTAs to the data with a pole fixed at that mass. The corresponding sequence will be called T_1^L . This has the obvious advantage that the number of parameters in the fit decreases by one and allows one to go a little further in the sequence. Our best value is then given by the Padé Type Approximant T_1^5 , whose expansion around $Q^2 = 0$ yields the following values for the Taylor coefficients:

$$a_1 = 1.90 \pm 0.03 \text{ GeV}^{-2}, \quad a_2 = 3.28 \pm 0.09 \text{ GeV}^{-4}, \quad (11)$$

with a $\chi^2/\text{dof} = 118/90$.

The previous analysis of PTAs may be extended by making further use of our knowledge of the vector spectroscopy [25]. For instance, by taking $M_\rho = 775.5$ MeV, $M_{\rho'} = 1459$ MeV and $M_{\rho''} = 1720$ MeV,⁴ we may construct further Padé-Type sequences of the form T_2^L and T_3^L .

In the PTA sequence T_2^L one needs to provide the value of two poles. For the first pole, the natural choice is M_ρ^2 . For the second pole, we found that choosing either M_ρ^2 or $M_{\rho''}^2$ (the second vector excitation) does not make any difference. Both outcomes are compared in Fig. (5). Using $M_{\rho''}^2$, we found that the T_2^3 PTA yields the best values as

$$a_1 = 1.902 \pm 0.024 \text{ GeV}^{-2}, \quad a_2 = 3.29 \pm 0.07 \text{ GeV}^{-4}, \quad (12)$$

with a $\chi^2/\text{dof} = 118/92$.

Using M_ρ^2 as the second pole one also gets the best value from the T_2^3 PTA, with the following results:

$$a_1 = 1.899 \pm 0.023 \text{ GeV}^{-2}, \quad a_2 = 3.27 \pm 0.06 \text{ GeV}^{-4}, \quad (13)$$

⁴As will be seen, results do not depend on the precise value chosen for these masses.

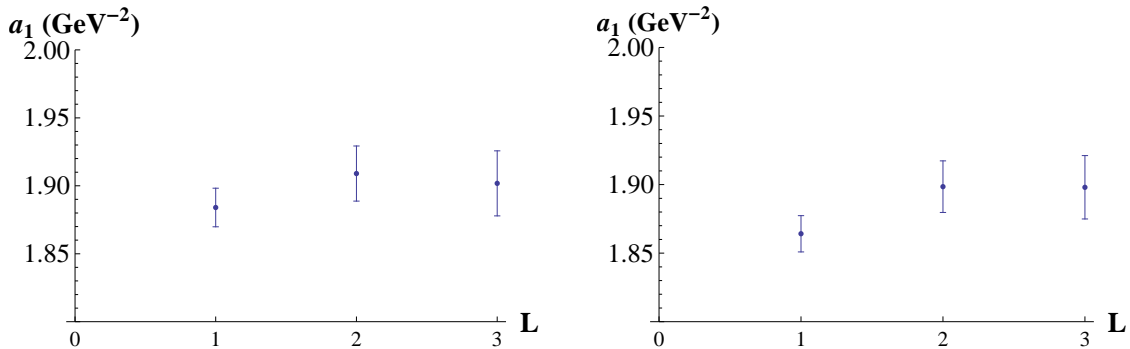


Figure 5: Low energy coefficient a_1 for the T_2^L Padé-Type sequence with M_ρ and $M_{\rho'}$ (left), and with M_ρ and $M_{\rho''}$ (right).

with a $\chi^2/\text{dof} = 119/92$. We find the stability of the results for the coefficients $a_{1,2}$ quite reassuring.

We have also performed an analysis of the PTA sequence T_3^L , with similar conclusions. From the T_3^3 we obtain the following values for the coefficients:

$$a_1 = 1.904 \pm 0.023 \text{ GeV}^{-2}, \quad a_2 = 3.29 \pm 0.09 \text{ GeV}^{-4}, \quad (14)$$

with a $\chi^2/\text{dof} = 119/92$.

Finally, to complete our analysis, we will also consider Partial Padé Approximants, in which only part of the denominator is given in advance. In particular, we study the PPA sequence $P_{1,1}^L$ ⁵ in which the first pole is given by M_ρ^2 and the other is left free. The best determination of the Taylor coefficients is given by $P_{1,1}^2$, and they yield

$$a_1 = 1.902 \pm 0.029 \text{ GeV}^{-2}, \quad a_2 = 3.28 \pm 0.09 \text{ GeV}^{-4}, \quad (15)$$

with the free pole of the PPA given by $M_{free}^2 = (1.6 \pm 0.4 \text{ GeV})^2$ and a $\chi^2/\text{dof} = 119/92$.

4 Combined Results and conclusions

Combining all the previous rational approximants results in an average given by

$$a_1 = 1.907 \pm 0.010_{\text{stat}} \pm 0.03_{\text{syst}} \text{ GeV}^{-2}, \quad a_2 = 3.30 \pm 0.03_{\text{stat}} \pm 0.33_{\text{syst}} \text{ GeV}^{-4}. \quad (16)$$

The first error comes from combining the results of the different fits by means of a weighted average. On top of that, we have added what we believe to be a conservative estimate of the theoretical (i.e. systematic) error based on the analysis of the VFF model in Sec. 2. We expect the latter to give an estimate for the systematic uncertainty due to the approximation of the physical form factor with rational functions. For comparison with previous analyses, we also provide in Table 2 the value of the quadratic radius, which is given by $\langle r^2 \rangle = 6 a_1$.

⁵See Ref. [13] for notation.

	$\langle r^2 \rangle$ (fm ²)	a_2 (GeV ⁻⁴)
This work	$0.445 \pm 0.002_{\text{stat}} \pm 0.007_{\text{syst}}$	$3.30 \pm 0.03_{\text{stat}} \pm 0.33_{\text{syst}}$
CGL [4, 5]	0.435 ± 0.005	...
TY [7]	0.432 ± 0.001	3.84 ± 0.02
BCT [27]	0.437 ± 0.016	3.85 ± 0.60
PP [9]	0.430 ± 0.012	3.79 ± 0.04
Lattice [28]	0.418 ± 0.031	...

Table 2: Our results for the quadratic radius $\langle r^2 \rangle$ and second derivative a_2 are compared to other determinations [4, 5, 7, 27, 9, 28]. Our first error is statistical. The second one is systematic, based on the analysis of the VFF model in section 2.

In summary, in this work we have used rational approximants as a tool for fitting the pion vector form factor. Because these approximants are capable of describing the region of large momentum, they may be better suited than polynomials for a description of the space-like data. As our results in Table 2 show, the errors achieved with these approximants are competitive with previous analyses existing in the literature.

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